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LETTER TO THE EDITOR

A generalized nonclassical state of the radiation field and some of its properties

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Abstract. We construct a generalized nonclassical state of the quantized radiation field (which in different limits yields the vacuum state, the number state, the binomial state, the negative binomial state and the coherent state) and examine various nonclassical properties of this state.

Nonclassical states of the radiation field have been an active field of research for the last few years because they not only reveal the quantum essence of the radiation field but also have potential application in optics communication, detection of weak signals, atomic and molecular physics etc. These nonclassical states are usually constructed from the number state $|n\rangle$ [1] or the coherent state $|\alpha\rangle$ [2]. Among the various nonclassical states, the binomial state [3] and the negative binomial state [4] occupy a special position in the sense that they are intermediate states. The binomial state is intermediate between the coherent state $|\alpha\rangle$ and the number state $|n\rangle$ and the negative binomial state is intermediate between the thermal state and the coherent state. Both of these states have been shown to exhibit nonclassical properties, such as antibunching, sub-Poissonian statistics and squeezing [3–5].

In this letter, we shall study a generalized nonclassical state which in different limits yields the vacuum state, the number state, the binomial state, the negative binomial state and the coherent state. In particular, we shall study the various nonclassical properties exhibited by them.

First, we note that the binomial state is defined as [3]

$$|p, M\rangle = \sum_{n=0}^{N} B_n^M |p\rangle = \sum_{n=0}^{M} \sqrt{{}^M C_n p^n (1-p)^{M-n}} |n\rangle \qquad 0 \leqslant p \leqslant 1. \tag{1}$$

From (1) it follows that the square of the weight associated with the number state $|n\rangle$ is the binomial distribution. In this context it may be noted that the binomial distribution acts as the weight function with respect to which the Kravchuk polynomials are orthogonal [6, 7]. Similarly, in the case of a coherent state the square of the weight associated with the number state acts as the weight function with respect to which the Charlier polynomials are orthogonal [6, 7]. This, in fact, has motivated us to look for more general orthogonal polynomials which yield all other orthogonal polynomials in different limits. One such polynomial is the Hahn polynomial [7, 8] which contains all other classical orthogonal

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polynomials as limiting cases. Hahn polynomials are orthogonal polynomials of a discrete variable and measure of orthogonality corresponding to these polynomials is given by [9]

$$\omega(n, N, \alpha, \beta) = \lambda \frac{(\alpha+1)_n(\beta+1)_{N-n}}{n!(N-n)!} \qquad n = 0, 1, \ldots, N \tag{2}$$

$$\lambda = \frac{N!}{(\alpha+\beta+2)_N} \qquad \alpha, \beta > -1 \tag{3}$$

where

$$(a)_0 = 1 \qquad (a)_n = a(a+1)\cdots(a+n-1).$$

Then using the result

$$\sum_{n=0}^{N} \frac{(a)_n(b)_{N-n}}{n!(N-n)!} = \frac{(a+b)_N}{N!} \tag{4}$$

it can be shown that $\omega(n, N, \alpha, \beta)$ is indeed a normalized probability measure:

$$\sum_{n=0}^{N} \omega(n, N, \alpha, \beta) = 1. \tag{5}$$

Now following [3] we define the generalized nonclassical state $|N, \alpha, \beta\rangle$ as

$$|N, \alpha, \beta\rangle = \sum_{n=0}^{N} \sqrt{\omega(n, N, \alpha, \beta)}|n\rangle \tag{6}$$

where $|n\rangle$ denotes the harmonic oscillator eigenkets (number states).

We can now verify that

(i) as $\alpha \to -1$, $|N, \alpha, \beta\rangle \to |0\rangle$, the vacuum state;

(ii) as $\beta \to -1$ and $\alpha = 0$, $|N, \alpha, \beta\rangle \to |N\rangle$, the number state;

(iii) as $\alpha \to \infty$ with $\beta = \frac{\alpha(1-p)}{p}$, $0 < p < 1$, $|\alpha, \beta, N\rangle \to |N, p\rangle$ where $|N, p\rangle$ is the binomial state;

(iv) as $N \to \infty$ with $\alpha = \gamma - 1$ and $\beta = \frac{pN}{(1-p)}$, $0 < p < 1$, $|N, \alpha, \beta\rangle \to |\gamma, p\rangle$ where $|\gamma, p\rangle$ is the negative binomial state;

(v) as $N \to \infty$ with $\alpha = N - 1$, $\beta = |\frac{N^2}{\lambda^2}$, $|N, \alpha, \beta\rangle \to |\lambda\rangle$ where $|\lambda >$ is the coherent state.

We shall now examine various properties of the state $|N, \alpha, \beta\rangle$. To do this we shall need several expectation values and by using (4) and (6) it can be shown that

$$\langle aa^\dagger\rangle = 1 + \frac{N(\alpha+1)}{(\alpha+\beta+2)} \tag{7}$$

$$\langle a^\dagger a\rangle = \frac{N(\alpha+1)}{(\alpha+\beta+2)} \tag{8}$$

$$\langle a^2\rangle = \sqrt{\frac{N(N+1)(\alpha+1)(\alpha+2)}{(\alpha+\beta+2)(\alpha+\beta+3)}} \sum_{n=0}^{N-2} \sqrt{\omega(n, N, \alpha, \beta)\omega(n, N-2, \alpha+2, \beta} = \langle a^{\dagger 2}\rangle \tag{9}$$

$$\langle a\rangle = \langle a^\dagger\rangle = \frac{N(\alpha+1)}{(\alpha+\beta+2)} \sum_{n=0}^{N-1} \sqrt{\omega(n, N-1, \alpha+1, \beta)} \tag{10}$$

$$\langle a^{\dagger 2}a^2\rangle = \frac{N(N-1)(\alpha+1)(\alpha+2)}{(\alpha+\beta+2)(\alpha+\beta+3)} \tag{11}$$

$$\langle a^\dagger a a^\dagger a\rangle = \frac{N(\alpha+1)}{(\alpha+\beta+2)} + \frac{N(N-1)(\alpha+1)(\alpha+2)}{(\alpha+\beta+2)(\alpha+\beta+3)}. \tag{12}$$

We shall now calculate the Mandel parameter [10]. The Mandel parameter Q is defined by

$$Q = \frac{\langle a^\dagger a a^\dagger a\rangle - \langle a^\dagger a\rangle^2 - \langle a^\dagger a\rangle}{\langle a^\dagger a\rangle}. \tag{13}$$

Then using (9) and (12) we find

$$Q = \frac{N(\beta+1)}{(\alpha+\beta+2)(\alpha+\beta+3)} - \frac{(\alpha+2)}{(\alpha+\beta+3)}. \tag{14}$$

It may be recalled that $Q > 1$ implies super-Poissonian, $Q = 1$ Poissonian and $Q < 1$ sub-Poissonian statistics. Therefore from (14) we conclude that the state $|N, \alpha, \beta\rangle$ exhibits sub-Poissonian statistics.

Let us now examine the antibunching effect for states $|N, \alpha, \beta\rangle$. The relevant parameter is given by

$$g^2(0) = \frac{a^{\dagger 2} a^2}{|\langle a^\dagger a\rangle|^2}. \tag{15}$$

Using (9) and (11) it follows from (15) that

$$g^2(0) = \frac{(N-1)(\alpha+2)(\alpha+\beta+2)}{N(\alpha+1)(\alpha+\beta+3)} \tag{16}$$

so that

$$g^2(0) < 1 \Rightarrow N < \frac{(\alpha+2)(\alpha+\beta+2)}{(\beta+1)}. \tag{17}$$

Thus for a given value of N we can always find values of α and β such that condition (17) is satisfied, i.e. the state $|N, \alpha, \beta\rangle$ exhibits the antibunching effect. On the other hand, when α and β are prescribed it is possible to find a value of $N = N_0$ such that antibunching takes place for $N < N_0$.

Finally we shall examine squeezing properties of the state $|N, \alpha, \beta\rangle$. To determine the squeezing effect we first define a pair of canonical quadrature operators:

$$X_1 = \frac{a + a^\dagger}{2} \qquad X_2 = \frac{a - a^\dagger}{2i} \tag{18}$$

which obey the uncertainty relation

$$\langle(\Delta X_1)^2\rangle\langle(\Delta X_2)^2\rangle \geqslant \tfrac{1}{16} \tag{19}$$

where $\langle(\Delta X)^2\rangle = \langle X^2\rangle - \langle X\rangle^2$ and the field is said to be squeezed if

$$\langle(\Delta A)^2\rangle < \tfrac{1}{4} \tag{20}$$

where $A = X_1$ or X_2. Now using (7)–(10) we find

$$\langle(\Delta X_1)^2\rangle - \tfrac{1}{4} = f + g - h \tag{21}$$

and

$$\langle(\Delta X_2)^2\rangle - \tfrac{1}{4} = f - g \tag{22}$$

where

$$f = \frac{2N(\alpha+1)}{(\alpha+\beta+2)} \tag{23}$$

$$g = 2\sqrt{\frac{N(N-1)(\alpha+1)(\alpha+2)}{(\alpha+\beta+2)(\alpha+\beta+3)}} \sum_{n=0}^{N-2} \sqrt{\omega(n, N, \alpha, \beta)\omega(n, N-2, \alpha+2, \beta)} \tag{24}$$

$$h = 4\frac{N(\alpha+1)}{(\alpha+\beta+1)} \left[\sum_{n=0}^{N-1} \sqrt{\omega(n, N, \alpha, \beta)\omega(n, N-1, \alpha+1, \beta)}\right]^2. \tag{25}$$

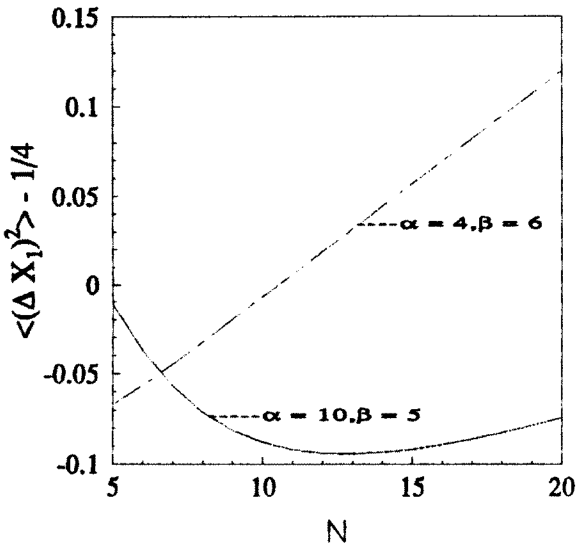

Figure 1. Plot of $\langle(\Delta X_1))^2\rangle - \frac{1}{4}$ against N for $\alpha = 4$, $\beta = 6$ and $\alpha = 10$, $\beta = 5$.

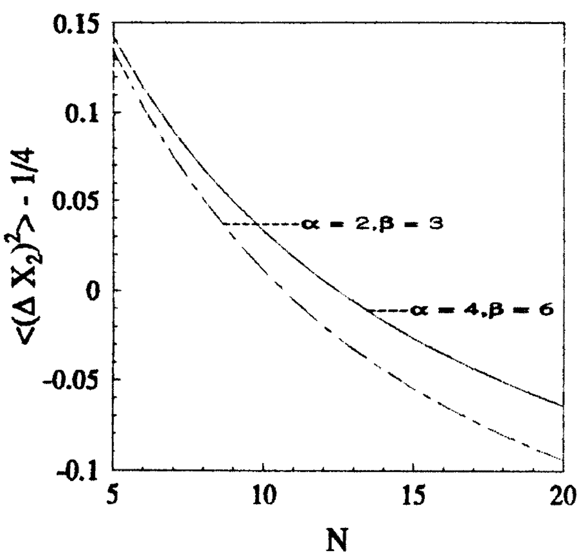

Figure 2. Plot of $\langle\Delta X_2)^2\rangle - \frac{1}{4}$ against N for $\alpha = 4$, $\beta = 6$ and $\alpha = 2$, $\beta = 3$.

It is now necessary to verify the squeezing conditions (20) with given by (23) and (24). In our numerical study we have plotted the two quadrature variances against N for fixed values of α and β as shown in figures 1 and 2 respectively. From the figures it is clear that there can be squeezing in either of the components.

In summary we have constructed a generalized nonclassical state of the radiation field which in different limits yields various other states. It has also been shown that this state exhibits sub-Poissonian behaviour, antibunching and quadrature squeezing.

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